

14.452 Economic Growth: Lectures 1 (second half), 2 and 3

The Solow Growth Model

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Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar model emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical* aggregate production function.

Households and Production I

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by $t = 0, 1, 2, \dots$
- Economy is inhabited by a large number of households, and for now households will not be optimizing.
- This is the main difference between the Solow model and the *neoclassical growth model*.
- To fix ideas, assume all households are identical, so the economy admits *a representative household*.

Households and Production II

- Assume households save a constant exogenous fraction s of their disposable income
- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \quad (1)$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$ is a *shifter* of the production function (1). Broad notion of technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good.

Key Assumption

Assumption 1 (Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale) The production function $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is twice continuously differentiable in K and L , and satisfies

$$\begin{aligned} F_K(K, L, A) &\equiv \frac{\partial F(\cdot)}{\partial K} > 0, & F_L(K, L, A) &\equiv \frac{\partial F(\cdot)}{\partial L} > 0, \\ F_{KK}(K, L, A) &\equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, & F_{LL}(K, L, A) &\equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0. \end{aligned}$$

Moreover, F exhibits constant returns to scale in K and L .

- Assume F exhibits *constant returns to scale* in K and L . I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition Let K be an integer. The function $g : \mathbb{R}^{K+2} \rightarrow \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

Theorem (Euler's Theorem) Suppose that $g : \mathbb{R}^{K+2} \rightarrow \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y . Then

$$\begin{aligned} mg(x, y, z) &= g_x(x, y, z)x + g_y(x, y, z)y \\ \text{for all } x &\in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K. \end{aligned}$$

Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogeneous of degree $m - 1$ in x and y .

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy, $\bar{L}(t)$, and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t)$$

for all t , where $L(t)$ denotes the demand for labor (and also the level of employment).

- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time t be $w(t)$, then the labor market clearing condition takes the form

$$L(t) \leq \bar{L}(t), w(t) \geq 0 \text{ and } (L(t) - \bar{L}(t)) w(t) = 0$$

Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.
- Households also own the capital stock of the economy and rent it to firms.
- Denote the *rental price of capital* at time t be $R(t)$.
- Capital market clearing condition:

$$K^s(t) = K^d(t)$$

- Take households' initial holdings of capital, $K(0)$, as given
- $P(t)$ is the price of the final good at time t , normalize the price of the final good to 1 *in all periods*.
- Build on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another.

Market Structure, Endowments and Market Clearing III

- Implies that we need to keep track of an *interest rate* across periods, $r(t)$, and this will enable us to normalize the price of the final good to 1 in every period.
- *General equilibrium economies*, where different commodities correspond to the same good at different dates.
- The same good at different dates (or in different states or localities) is a different commodity.
- Therefore, there will be *an infinite number of commodities*.
- Assume capital depreciates, with “exponential form,” at the rate δ : out of 1 unit of capital this period, only $1 - \delta$ is left for next period.
- Loss of part of the capital stock affects the interest rate (rate of return to savings) faced by the household.
- *Interest rate* faced by the household will be $r(t) = R(t) - \delta$.

Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t) \geq 0, K(t) \geq 0} F[K(t), L(t), A(t)] - w(t) L(t) - R(t) K(t).$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem*.
- Features worth noting:
 - ① Problem is set up in terms of aggregate variables.
 - ② Nothing multiplying the F term, price of the final good has normalized to 1.
 - ③ Already imposes competitive factor markets: firm is taking as given $w(t)$ and $R(t)$.
 - ④ Concave problem, since F is concave.

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)], \quad (2)$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \quad (3)$$

- Note also that in (2) and (3), we used $K(t)$ and $L(t)$, the amount of capital and labor used by firms.
- In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $R(t)$ and $w(t)$.
- Thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

Firm Optimization III

Proposition Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t)L(t) + R(t)K(t).$$

- **Proof:** Follows immediately from Euler Theorem for the case of $m = 1$, i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

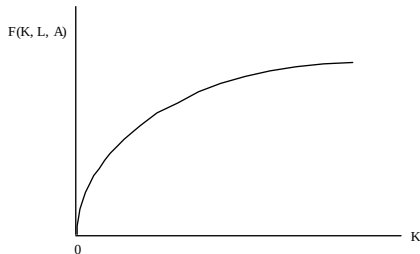
Second Key Assumption

Assumption 2 (Inada conditions) F satisfies the Inada conditions

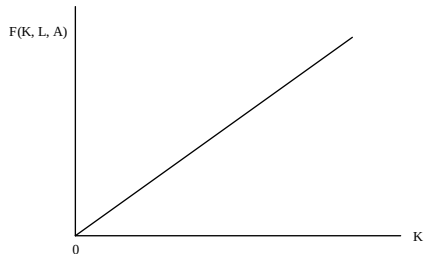
$$\begin{aligned}\lim_{K \rightarrow 0} F_K(\cdot) &= \infty \text{ and } \lim_{K \rightarrow \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L \rightarrow 0} F_L(\cdot) &= \infty \text{ and } \lim_{L \rightarrow \infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A.\end{aligned}$$

- Important in ensuring the existence of *interior equilibria*.
- It can be relaxed quite a bit, though useful to get us started.

Production Functions



Panel A



Panel B

Figure: Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ , so

$$K(t+1) = (1 - \delta) K(t) + I(t), \quad (4)$$

where $I(t)$ is investment at time t .

- From national income accounting for a closed economy,

$$Y(t) = C(t) + I(t), \quad (5)$$

- Using (1), (4) and (5), any *feasible* dynamic allocation in this economy must satisfy

$$K(t+1) \leq F[K(t), L(t), A(t)] + (1 - \delta) K(t) - C(t)$$

for $t = 0, 1, \dots$

- Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.

Fundamental Law of Motion of the Solow Model II

- Note not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.
- Since the economy is closed (and there is no government spending),

$$S(t) = I(t) = Y(t) - C(t).$$

- Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \quad (6)$$

$$C(t) = (1 - s)Y(t) \quad (7)$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s(t) = (1 - \delta)K(t) + S(t) = (1 - \delta)K(t) + sY(t).$$

Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies $K^s(t) = K(t)$.
- We also have $L(t) = \bar{L}(t)$.
- Combining these market clearing conditions with (1) and (4), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1 - \delta)K(t). \quad (8)$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

Definition of Equilibrium I

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.

Definition In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies (8), $Y(t)$ is given by (1), $C(t)$ is given by (7), and $w(t)$ and $R(t)$ are given by (2) and (3).

- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.

Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
 - ① There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L(t) = L$.
 - ② No technological progress, so that $A(t) = A$.
- Define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \quad (9)$$

- Using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t) / L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \quad (10)$$

Equilibrium Without Population Growth and Technological Progress II

- Note that $f(k)$ here depends on A , so I could have written $f(k, A)$; but A is constant and can be normalized to $A = 1$.
- From Euler Theorem,

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t) f'(k(t)) > 0. \end{aligned} \tag{11}$$

- Both are positive from Assumption 1.

Example: The Cobb-Douglas Production Function I

- Very special production function but widely used:

$$\begin{aligned} Y(t) &= F[K(t), L(t), A(t)] \\ &= AK(t)^\alpha L(t)^{1-\alpha}, \quad 0 < \alpha < 1. \end{aligned}$$

- Satisfies Assumptions 1 and 2.
- Dividing both sides by $L(t)$,

$$y(t) = Ak(t)^\alpha,$$

- From equation (11),

$$R(t) = \frac{\partial Ak(t)^\alpha}{\partial k(t)} = \alpha Ak(t)^{-(1-\alpha)}.$$

- From the Euler Theorem,

$$w(t) = y(t) - R(t)k(t) = (1 - \alpha)Ak(t)^\alpha.$$

Example: The Cobb-Douglas Production Function II

- Alternatively, in terms of the original Cobb-Douglas production function,

$$\begin{aligned} R(t) &= \alpha A K(t)^{\alpha-1} L(t)^{1-\alpha} \\ &= \alpha A k(t)^{-(1-\alpha)}, \end{aligned}$$

and similarly, from (11),

$$\begin{aligned} w(t) &= (1-\alpha) A K(t)^{\alpha} L(t)^{-\alpha} \\ &= (1-\alpha) A k(t)^{\alpha}, \end{aligned}$$

verifying the Euler Theorem in this case.

Equilibrium Without Population Growth and Technological Progress I

- The per capita representation of the aggregate production function enables us to divide both sides of (8) by L to obtain:

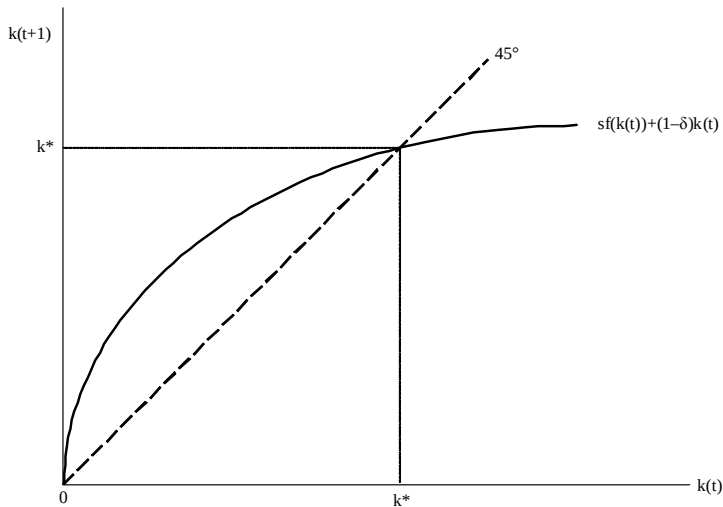
$$k(t+1) = sf(k(t)) + (1 - \delta)k(t). \quad (12)$$

- Since it is derived from (8), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

Definition A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t .

- The economy will tend to this steady state equilibrium over time (but never reach it in finite time).

Steady-State Capital-Labor Ratio



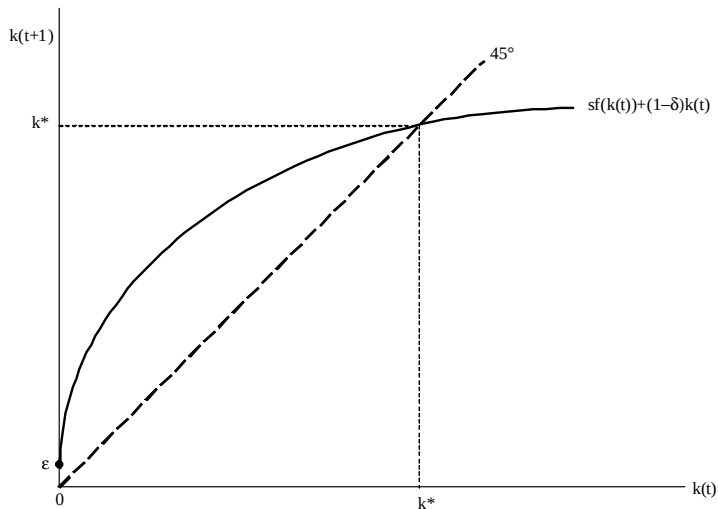
Equilibrium Without Population Growth and Technological Progress II

- Thick curve represents (12) and the dashed line corresponds to the 45° line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^* ,

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \quad (13)$$

- There is another intersection at $k = 0$, because the figure assumes that $f(0) = 0$.
- Will ignore this intersection throughout:
 - If capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium
 - This intersection, even when it exists, is an *unstable point*
 - It has no economic interest for us.

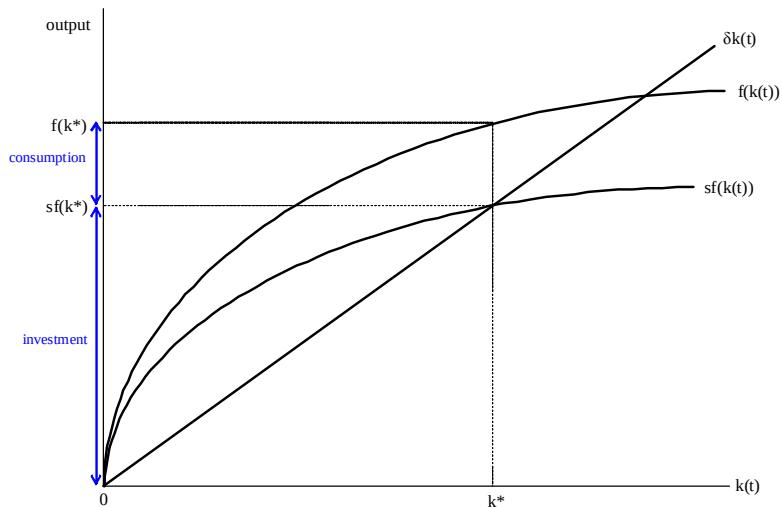
Equilibrium Without Population Growth and Technological Progress III



Equilibrium Without Population Growth and Technological Progress IV

- Alternative visual representation of the steady state: intersection between δk and the function $sf(k)$. Useful because:
 - 1 Depicts the levels of consumption and investment in a single figure.
 - 2 Emphasizes the steady-state equilibrium sets investment, $sf(k)$, equal to the amount of capital that needs to be “replenished”, δk .

Consumption and Investment in Steady State



Equilibrium Without Population Growth and Technological Progress V

Proposition Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio $k^* \in (0, \infty)$ is given by (13), per capita output is given by

$$y^* = f(k^*) \quad (14)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \quad (15)$$

Proof

- The preceding argument establishes that any k^* that satisfies (13) is a steady state.
- To establish existence, note that from Assumption 2 (and from L'Hospital's rule), $\lim_{k \rightarrow 0} f(k)/k = \infty$ and $\lim_{k \rightarrow \infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous from Assumption 1, so by the Intermediate Value Theorem there exists k^* such that (13) is satisfied.
- To see uniqueness, differentiate $f(k)/k$ with respect to k , which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \quad (16)$$

where the last equality uses (11).

- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (13).
- Equations (14) and (15) then follow by definition.

Non-Existence and Non-Uniqueness

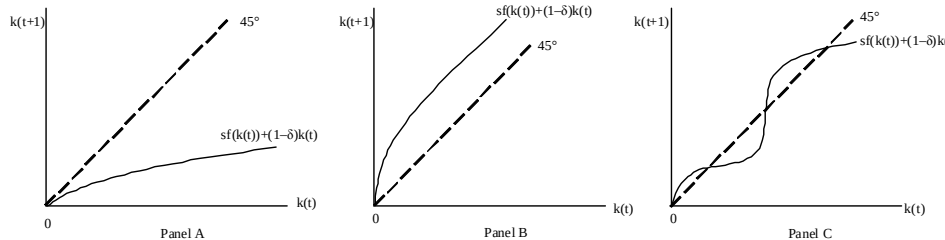


Figure: Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VI

- Comparative statics with respect to s , a and δ straightforward for k^* and y^* .
- But c^* will not be monotone in the saving rate (think, for example, of $s = 1$).
- In fact, there will exist a specific level of the saving rate, s_{gold} , referred to as the “golden rule” saving rate, which maximizes c^* .
- But cannot say whether the golden rule saving rate is “better” than some other saving rate.
- Write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned}c^*(s) &= (1 - s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s),\end{aligned}$$

- The second equality exploits that in steady state $sf(k) = \delta k$.

Equilibrium Without Population Growth and Technological Progress X

- Differentiating with respect to s ,

$$\frac{\partial c^*(s)}{\partial s} = [f'(k^*(s)) - \delta] \frac{\partial k^*}{\partial s}. \quad (17)$$

- s_{gold} is such that $\partial c^*(s_{gold}) / \partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k_{gold}^* .

Proposition In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold} , with the corresponding steady state capital level k_{gold}^* such that

$$f'(k_{gold}^*) = \delta. \quad (18)$$

The Golden Rule

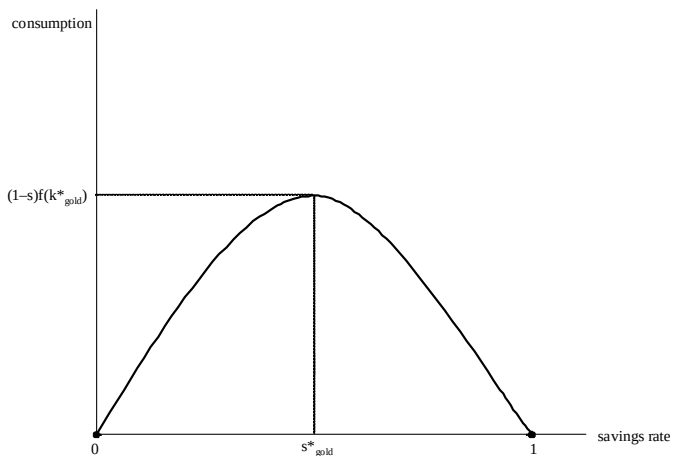


Figure: The “golden rule” level of savings rate, which maximizes steady-state consumption.

Dynamic Inefficiency

- When the economy is below k_{gold}^* , higher saving will increase consumption; when it is above k_{gold}^* , steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency*).
- But no utility function, so statements about “inefficiency” have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.

Summing up: the Discrete-Time Solow Model

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1-\delta)k(t).$$

- The steady-state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

- Consumption is given by

$$C(t) = (1-s)Y(t)$$

- And factor prices are given by

$$R(t) = f'(k(t)) > 0 \text{ and}$$

$$w(t) = f(k(t)) - k(t)f'(k(t)) > 0.$$

Steady State Equilibrium

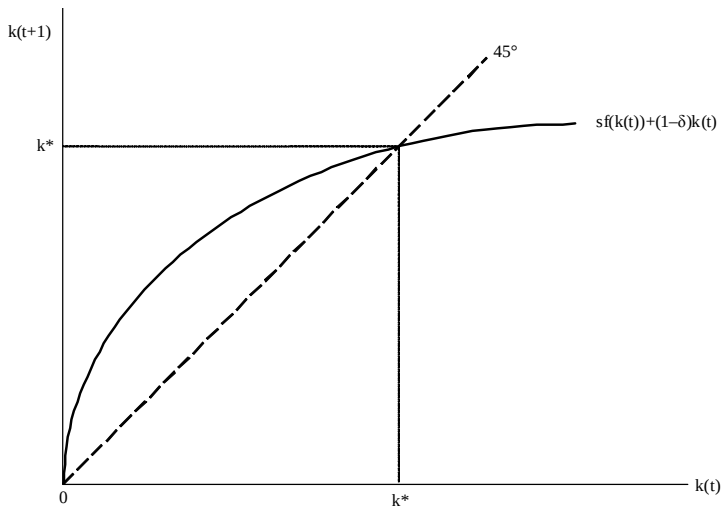


Figure: Steady-state capital-labor ratio in the Solow model.

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
 - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
 - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the “transitional dynamics” of the equilibrium difference equation (12) starting from an arbitrary initial capital-labor ratio $k(0) > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \quad (19)$$

- $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$ is sometimes referred to as “an equilibrium point” of (19).
- We will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (19).

Definition A steady state $\mathbf{x}^*$ is (locally) *asymptotically stable* if there exists an open set $B(\mathbf{x}^*) \ni \mathbf{x}^*$ such that for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ to (19) with $\mathbf{x}(0) \in B(\mathbf{x}^*)$, we have $\mathbf{x}(t) \rightarrow \mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is *globally asymptotically stable* if for all $\mathbf{x}(0) \in \mathbb{R}^n$, for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, we have $\mathbf{x}(t) \rightarrow \mathbf{x}^*$.

Transitional Dynamics: Review II

Simple Result About Stability

- Let $x(t)$, $a, b \in \mathbb{R}$, then the unique steady state of the linear difference equation $x(t+1) = ax(t) + b$ is globally asymptotically stable (in the sense that $x(t) \rightarrow x^* = b/(1-a)$) if $|a| < 1$.
- Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at the steady state x^* , defined by $g(x^*) = x^*$. Then, the steady state of the nonlinear difference equation $x(t+1) = g(x(t))$, x^* , is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Transitional Dynamics in the Discrete Time Solow Model

Proposition Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (12) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^* .

Proof of Proposition: Transitional Dynamics I

- Let $g(k) \equiv sf(k) + (1 - \delta)k$. First observe that $g'(k) > 0$ for all k .
- Next, from (12),

$$k(t+1) = g(k(t)), \quad (20)$$

with a unique steady state at k^* .

- From (13), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or

$$k^* = g(k^*). \quad (21)$$

- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2.

Proof of Proposition: Transitional Dynamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \quad (22)$$

- The second inequality uses the fact that $f(0) \geq 0$.
- Since (22) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dynamics III

- To prove global stability, note that for all $k(t) \in (0, k^*)$,

$$\begin{aligned}k(t+1) - k^* &= g(k(t)) - g(k^*) \\&= - \int_{k(t)}^{k^*} g'(k) dk, \\&< 0\end{aligned}$$

- First line follows by subtracting (21) from (20), second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'(k) > 0$ for all k .

Proof of Proposition: Transitional Dynamics IV

- Next, (12) also implies

$$\begin{aligned}\frac{k(t+1) - k(t)}{k(t)} &= s \frac{f(k(t))}{k(t)} - \delta \\ &> s \frac{f(k^*)}{k^*} - \delta \\ &= 0.\end{aligned}$$

Moreover, for any $k(t) \in (0, k^* - \varepsilon)$, this is uniformly so.

- Second line uses the fact that $f(k)/k$ is decreasing in k (from (22) above) and last line uses the definition of k^* .
- These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$.
- An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$.
- Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics III

- Stability result can be seen diagrammatically in the Figure:
 - Starting from initial capital stock $k(0) < k^*$, economy grows towards k^* , *capital deepening* and growth of per capita income.
 - If economy were to start with $k'(0) > k^*$, reach the steady state by decumulating capital and contracting.
- As a consequence:

Proposition Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k(0) < k^*$.

Transitional Dynamics in Figure

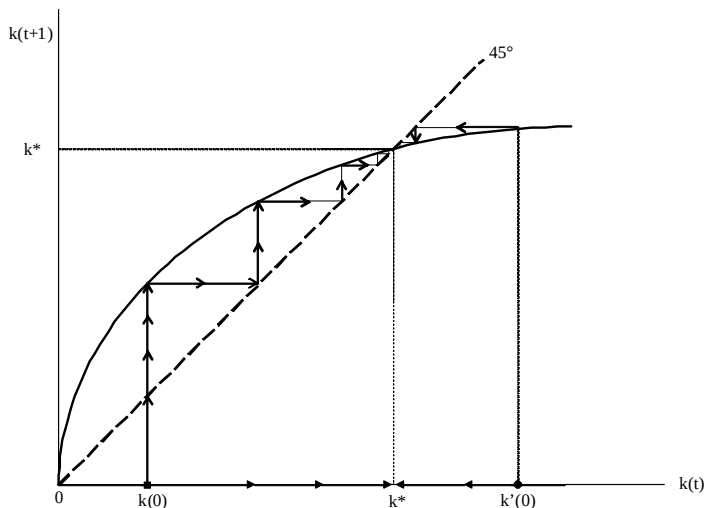


Figure: Transitional dynamics in the basic Solow model.

From Difference to Differential Equations I

- Start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)). \quad (23)$$

- Now consider the following approximation for any $\Delta t \in [0, 1]$,

$$x(t + \Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

- When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives (23).
- In-between it is a linear approximation, not too bad if $g(x) \simeq g(x(t))$ for all $x \in [x(t), x(t+1)]$

From Difference to Differential Equations II

- Divide both sides of this equation by Δt , and take limits

$$\lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \quad (24)$$

where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (24) is a differential equation representing (23) for the case in which t and $t + 1$ is “small”.

The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, so (11) still give the factor prices, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S(t) = sY(t),$$

- Consumption is given by (7) above.
- Introduce population growth,

$$L(t) = \exp(nt) L(0). \quad (25)$$

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

The Fundamental Equation of the Solow Model in Continuous Time II

- Implies

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n.\end{aligned}$$

- From the limiting argument leading to equation (24),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of $k(t)$ and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s \frac{f(k(t))}{k(t)} - (n + \delta), \quad (26)$$

The Fundamental Equation of the Solow Model in Continuous Time III

Definition In the basic Solow model in continuous time with population growth at the rate n , no technological progress and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $L(t)$ satisfies (25), $k(t) \equiv K(t)/L(t)$ satisfies (26), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (7), and $w(t)$ and $R(t)$ are given by (11).

- As before, *steady-state* equilibrium involves $k(t)$ remaining constant at some level k^* .

Steady State With Population Growth

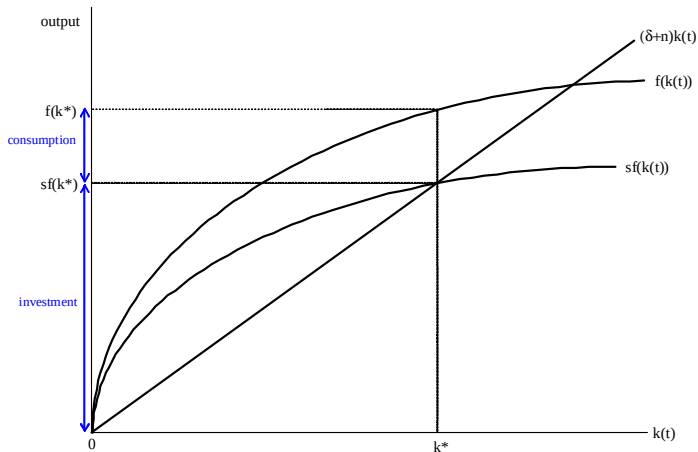


Figure: Investment and consumption in the steady-state equilibrium with population growth.

Steady State of the Solow Model in Continuous Time

- Equilibrium path (26) has a unique *steady state* at k^* , which is given by a slight modification of (13) above:

$$\frac{f(k^*)}{k^*} = \frac{n + \delta}{s}. \quad (27)$$

Proposition Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (27), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*).$$

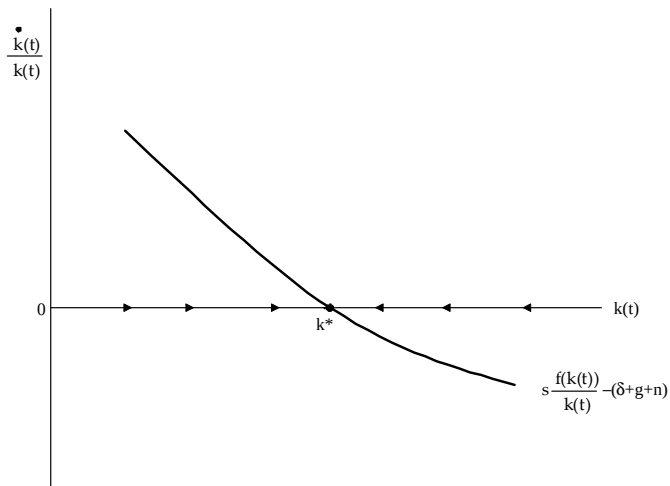
- Similar comparative statics to the discrete time model.

Transitional Dynamics in the Continuous Time Solow Model I

Simple Result about Stability In Continuous Time Model

- Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function and suppose that there exists a unique x^* such that $g(x^*) = 0$. Moreover, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^* , is globally asymptotically stable, i.e., starting with any $x(0)$, $x(t) \rightarrow x^*$.

Simple Result in Figure



Transitional Dynamics in the Continuous Time Solow Model II

Proposition Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \rightarrow k^*$.

- **Proof:** Follows immediately from the Theorem above by noting whenever $k < k^*$, $sf(k) - (n + \delta)k > 0$ and whenever $k > k^*$, $sf(k) - (n + \delta)k < 0$.
- Figure: plots the right-hand side of (26) and makes it clear that whenever $k < k^*$, $\dot{k} > 0$ and whenever $k > k^*$, $\dot{k} < 0$, so k monotonically converges to k^* .

A First Look at Sustained Growth I

- Cobb-Douglas already showed that when α is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes $\alpha = 1$ in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F[K(t), L(t), A(t)] = AK(t), \quad (28)$$

where $A > 0$ is a constant.

- So-called “AK” model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions,

$$F[K(t), L(t), A(t)] = AK(t) + BL(t), \quad (29)$$

A First Look at Sustained Growth II

- Assume population grows at n as before (cfr. equation (25)).
- Combining with the production function (28),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if $sA - \delta - n > 0$, there will be sustained growth in the capital-labor ratio.
- From (28), this implies that there will be sustained growth in output per capita as well.

A First Look at Sustained Growth III

Proposition Consider the Solow growth model with the production function (28) and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k(0) > 0$, the economy has

$$k(t) = \exp((sA - \delta - n)t) k(0)$$

and

$$y(t) = \exp((sA - \delta - n)t) A k(0).$$

- Note no transitional dynamics.

Sustained Growth in Figure

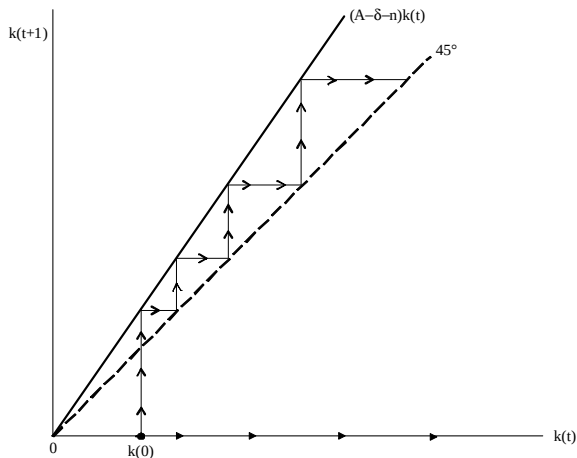


Figure: Sustained growth with the linear AK technology with $sA - \delta - n > 0$.

A First Look at Sustained Growth IV

- Unattractive features:

- ① Knife-edge case, requires the production function to be ultimately linear in the capital stock.
- ② Implies that as time goes by the share of national income accruing to capital will increase towards 1.
- ③ Technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth.

Balanced Growth I

- Production function $F[K(t), L(t), A(t)]$ is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963).
- Kaldor facts:
 - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.

Historical Factor Shares

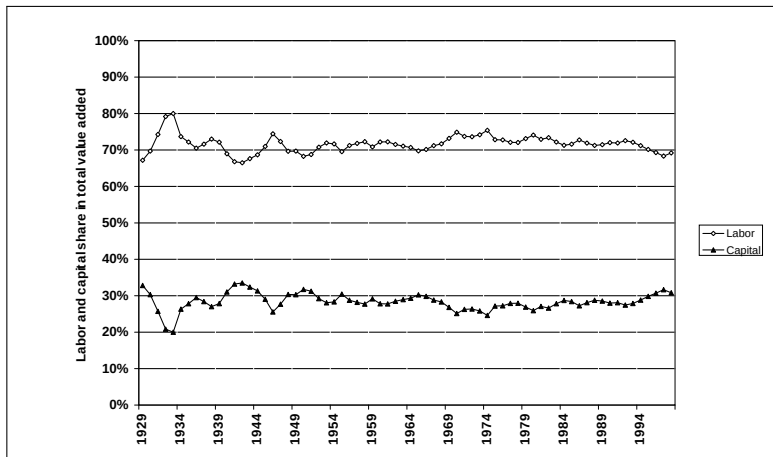


Figure: Capital and Labor Share in the U.S. GDP.

Balanced Growth II

- Note capital share in national income is about $1/3$, while the labor share is about $2/3$.
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose $AK^{1/3}L^{2/3}$.
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
 - i.e., we will again have $\dot{k} = 0$, but the definition of k will change.
- But important to bear in mind that growth has many non-balanced features.
 - e.g., the share of different sectors changes systematically.

Types of Neutral Technological Progress I

- For some constant returns to scale function $\tilde{F}$:

- *Hicks-neutral* technological progress:

$$\tilde{F}[K(t), L(t), A(t)] = A(t) F[K(t), L(t)],$$

- Relabeling of the isoquants (without any change in their shape) of the function $\tilde{F}[K(t), L(t), A(t)]$ in the L - K space.
 - *Solow-neutral* technological progress,

$$\tilde{F}[K(t), L(t), A(t)] = F[A(t) K(t), L(t)].$$

- Capital-augmenting progress: isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
 - *Harrod-neutral* technological progress,

$$\tilde{F}[K(t), L(t), A(t)] = F[K(t), A(t) L(t)].$$

- Increases output as if the economy had more labor: slope of the isoquants are constant along rays with constant capital-output ratio.

Isoquants with Neutral Technological Progress

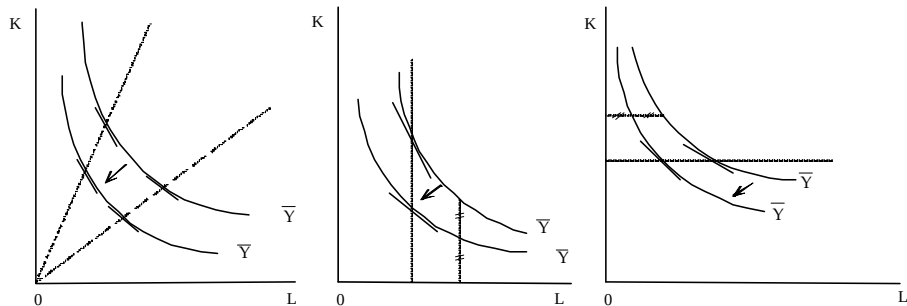


Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

Types of Neutral Technological Progress II

- Could also have a vector valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)],$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even this is a restriction on the form of technological progress, $A(t)$ could modify the entire production function.
- Balanced growth necessitates that all technological progress be labor augmenting or Harrod-neutral.

Uzawa's Theorem I

- Focus on continuous time models.
- Key elements of balanced growth: constancy of factor shares and of the capital-output ratio, $K(t) / Y(t)$.
- By factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t) L(t)}{Y(t)} \text{ and } \alpha_K(t) \equiv \frac{R(t) K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem $\alpha_L(t) + \alpha_K(t) = 1$.

Uzawa's Theorem II

Theorem

(Uzawa I) Suppose $\dot{L}(t) = \exp(nt) L(0)$,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$, and $\tilde{F}$ is CRS in K and L .

Suppose for $\tau < \infty$, $\dot{Y}(t)/Y(t) = g_Y > 0$, $\dot{K}(t)/K(t) = g_K > 0$ and $\dot{C}(t)/C(t) = g_C > 0$. Then,

- 1 $g_Y = g_K = g_C$; and
- 2 for any $t \geq \tau$, $\tilde{F}$ can be represented as

$$Y(t) = F(K(t), A(t)L(t)),$$

where $A(t) \in \mathbb{R}_+$, $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is homogeneous of degree 1, and

$$\dot{A}(t)/A(t) = g = g_Y - n.$$

Implications of Uzawa's Theorem

Corollary Under the assumptions of Uzawa Theorem, after time τ technological progress can be represented as Harrod neutral (purely labor augmenting).

- Remarkable feature: stated and proved without any reference to equilibrium behavior or market clearing.
- Also, contrary to Uzawa's original theorem, not stated for a balanced growth path but only for an asymptotic path with constant rates of output, capital and consumption growth.
- **But**, not as general as it seems;
 - the theorem gives only one representation.

Stronger Theorem

Theorem

(Uzawa's Theorem II) Suppose that all of the hypothesis in Uzawa's Theorem are satisfied, so that $\tilde{F} : \mathbb{R}_+^2 \times \mathcal{A} \rightarrow \mathbb{R}_+$ has a representation of the form $F(K(t), A(t)L(t))$ with $A(t) \in \mathbb{R}_+$ and $\dot{A}(t)/A(t) = g = g_Y - n$. In addition, suppose that factor markets are competitive and that for all $t \geq T$, the rental rate satisfies $R(t) = R^*$ (or equivalently, $\alpha_K(t) = \alpha_K^*$). Then, denoting the partial derivatives of $\tilde{F}$ and F with respect to their first two arguments by $\tilde{F}_K, \tilde{F}_L, F_K$ and F_L , we have

$$\begin{aligned}\tilde{F}_K(K(t), L(t), \tilde{A}(t)) &= F_K(K(t), A(t)L(t)) \text{ and} \\ \tilde{F}_L(K(t), L(t), \tilde{A}(t)) &= A(t) F_L(K(t), A(t)L(t)).\end{aligned}\quad (30)$$

Moreover, if (30) holds and factor markets are competitive, then $R(t) = R^*$ (and $\alpha_K(t) = \alpha_K^*$) for all $t \geq T$.

Intuition

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as $t \geq \tau$,

$$\begin{aligned}
 \alpha_K(t) &\equiv \frac{R(t) K(t)}{Y(t)} \\
 &= \frac{K(t)}{Y(t)} \frac{\partial F[K(t), A(t)L(t)]}{\partial K(t)} \\
 &= \alpha_K^*,
 \end{aligned}$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that $g_Y = g_K$ and $g_K = g + n$ from Uzawa Theorem and that F exhibits constant returns to scale so its derivative is homogeneous of degree 0.

Intuition for the Uzawa's Theorems

- We assumed the economy features capital accumulation in the sense that $g_K > 0$.
- From the aggregate resource constraint, this is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to n and there is no technological change (i.e., proposition applies with $g = 0$), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor.
- Constancy of growth requires technological change to make up for this asymmetry
- But this intuition does not provide a reason for why technology should take labor-augmenting (Harrod-neutral) form.
- But if technology did not take this form, an asymptotic path with constant growth rates would not be possible.

Interpretation

- Distressing result:
 - Balanced growth is only possible under a very stringent assumption.
 - Provides no reason why technological change should take this form.
- But when technology is endogenous, intuition above also works to make technology endogenously more labor-augmenting than capital augmenting.
- Not only requires labor augmenting asymptotically, i.e., along the balanced growth path.
- This is the pattern that certain classes of endogenous-technology models will generate.

Implications for Modeling of Growth

- Does not require $Y(t) = F[K(t), A(t)L(t)]$, but only that it has a representation of the form $Y(t) = F[K(t), A(t)L(t)]$.
- Allows one important exception. If,

$$Y(t) = [A_K(t)K(t)]^\alpha [A_L(t)L(t)]^{1-\alpha},$$

then both $A_K(t)$ and $A_L(t)$ could grow asymptotically, while maintaining balanced growth.

- Because we can define $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$ and the production function can be represented as

$$Y(t) = [K(t)]^\alpha [A(t)L(t)]^{1-\alpha}.$$

- Differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1.

Further Intuition

- Suppose the production function takes the special form $F[A_K(t) K(t), A_L(t) L(t)]$.
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when $A_K(t) K(t)$ and $A_L(t) L(t)$ grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that $K(t)$ must grow at the same rate as $A_L(t) L(t)$.
- Thus balanced growth can only be possible if $A_K(t)$ is asymptotically constant.

The Solow Growth Model with Technological Progress: Continuous Time I

- From Uzawa Theorem, production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \quad (31)$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \quad (32)$$

The Solow Growth Model with Technological Progress: Continuous Time II

- Now define $k(t)$ as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \quad (33)$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \quad (34)$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)). \end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is $y(t) \equiv Y(t) / L(t)$, i.e.,

$$\begin{aligned} y(t) &= A(t) \hat{y}(t) \\ &= A(t) f(k(t)). \end{aligned} \tag{35}$$

- Clearly if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will grow over time, since $A(t)$ is growing.
- Thus should not look for “steady states” where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as $\hat{y}(t)$ or $k(t)$ in (34) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms “steady state” and balanced growth path interchangeably.
- Substituting for $\dot{K}(t)$ from (32) into (34):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (33),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \quad (36)$$

- Only difference is the presence of g : k is no longer the capital-labor ratio but the *effective* capital-labor ratio.

The Solow Growth Model with Technological Progress: Continuous Time V

Proposition Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n . Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (33). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \quad (37)$$

Per capita output and consumption grow at the rate g .

The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (37), emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons:
 - 1 depreciation at the rate δ .
 - 2 population growth at the rate n , which reduces capital per worker.
 - 3 Harrod-neutral technological progress at the rate g .
- Now replenishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)k$.

The Solow Growth Model with Technological Progress: Continuous Time VII

Proposition Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k(t) \rightarrow k^*$).

- Now model generates growth in output per capita, but entirely *exogenously*.

Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall

$$\dot{k}(t) / k(t) = sf(k(t)) / k(t) - (\delta + g + n)$$

Comparative Dynamics in Figure

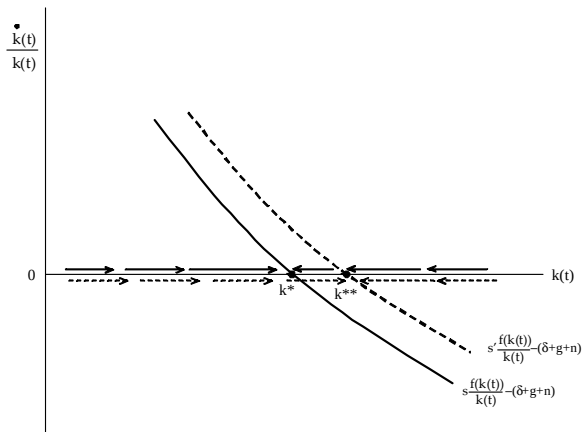


Figure: Dynamics following an increase in the savings rate from s to s' . The solid arrows show the dynamics for the initial steady state, while the dashed arrows

Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from s to s' .
 - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**} .
 - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**} .
 - Immediately, the capital stock remains unchanged (since it is a *state* variable).
 - After this point, it follows the dashed arrows on the horizontal axis.
- s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t'$.
 - Starting at t' , the economy follows the rightwards arrows until t' .
 - After t'' , the original steady state of the differential equation applies and leftwards arrows become effective.
 - From t'' onwards, economy gradually returns back to its original balanced growth equilibrium, k^* .

Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the AK world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a blackbox.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.